NON-OSCILLATORY LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society February 23, 1901.)

WE shall be concerned with the differential equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0, \tag{1}$$

and for the sake of simplicity we will assume that the coefficients p and q are, throughout the finite interval $a \leqq x \leqq b$, continuous real functions of the real variable x. We shall find it convenient to lay down the following definition:

The equation (1) is said to be oscillatory or non-oscillatory in the interval $a \leqq x \leqq b$ according as it does or does not have at least one solution (not identically zero) which vanishes more than once in this interval.

It is my object in the present paper to deduce certain conditions (chiefly sufficient conditions) that the equation (1) should be non-oscillatory. Such conditions have been obtained by Picard (Traité d'analyse, volume III, pp. 101–104); but the method which I use is not only entirely different and, as it seems to me, less artificial than that of Picard. but yields, besides all of Picard's results, others which Picard's method does not give.

My starting point is the special case $p = 0$:

$$\frac{d^2y}{dx^2} + qy = 0. \tag{2}$$

Equation (2) is non-oscillatory in the interval $a \leqq x \leqq b$ if throughout this interval $q \leqq 0$.

For if (2) has a solution y which vanishes more than once in the interval in question, let x_1 and x_2 be two successive roots of y. We may, without loss of generality, assume $y > 0$ when $x_1 < x < x_2$, as, if this were not the case, we could replace y by $y_1 = -y$. We have then $y'(x_1) > 0$, $y'(x_2) < 0$; but by the law of the mean

$$y'(x_2) - y'(x_1) = (x_2 - x_1)\, y''(\xi) \quad (x_1 < \xi < x_2).$$

Accordingly $y''(\xi) < 0$. This, however, is impossible, since by equation (2) $y''(\xi) = -q(\xi).y(\xi)$.

By reducing (1) to the binomial form (2), we shall get a theorem concerning (1) similar to the one just proved for (2). This reduction is most commonly performed by means of a change of dependent variable y.* It can, however, equally well be performed by changing the independent variable x, or, more generally, by changing both independent and dependent variable. We will consider at once this general transformation

$$t = f(x), \quad y = \varphi(x)\,\bar{y}. \tag{3}$$

We assume here that f and φ have continuous first and second derivatives throughout the interval $a \leqq x \leqq b$. Furthermore, since we do not wish the solutions of the transformed equation to become infinite, we assume that φ does not vanish, say for distinctness

$$\varphi(x) > 0 \qquad (a \leqq x \leqq b).$$

Finally, since we wish the interval $a \leqq x \leqq b$ to correspond in a one to one manner to an interval on the t-axis, we assume that f' does not vanish in the interval $a \leqq x \leqq b$.

A peculiarity of this transformation which makes it available for our purposes may be stated as follows:

The oscillatory or non-oscillatory character of equation (1) *is invariant with regard to transformations* (3).

The transformation (3) carries (1) over into

$$\frac{d^2\bar{y}}{dt^2} + \frac{1}{f'}\left[\frac{f''}{f'} + 2\frac{\varphi'}{\varphi} + p\right]\frac{d\bar{y}}{dt} + \frac{\varphi'' + p\varphi' + q\varphi}{\varphi \cdot (f')^2}\,\bar{y} = 0, \tag{4}$$

accents denoting differentiation with regard to x.

Choosing φ at pleasure, subject to the restrictions above mentioned, let us determine f so that the second term of (4) drops out,

$$f = k\int_c^x \varphi^{-2} e^{-\int_c^x p\,dx}\,dx + k' \qquad (k \neq 0), \tag{5}$$

where c is any point of the interval $a \leqq x \leqq b$.

It is important to notice that all the conditions which f was to fulfill are satisfied by this function.

Equation (4) now reduces to

*See, *e. g.*, Forsyth's Treatise on differential equations, p. 88. I note in passing that this reduction is possible only if the coefficient p has a continuous first derivative, a restriction which need not be imposed if we reduce to the binomial form by a change of independent variable.

$$(6) \qquad \frac{d^2\bar{y}}{dt^2} + \frac{\varphi^3}{k^2} e^{2\int_c^x p dx} (\varphi'' + p\varphi' + q\varphi)\bar{y} = 0.$$

When we apply to this equation the theorem proved concerning equation (2), we see that equation (6), and accordingly also equation (1), will be non-oscillatory provided $\varphi'' + p\varphi' + q\varphi \leqq 0$. That is

If a function φ exists which, together with its first and second derivatives, is continuous throughout the interval $a \leqq x \leqq b$, and which satisfies the two conditions

$$(7) \qquad \varphi > 0 \qquad (a \leqq x \leqq b),$$

$$(8) \qquad \varphi'' + p\varphi' + q\varphi \leqq 0 \qquad (a \leqq x \leqq b),$$

then (1) *is non-oscillatory in this interval.**

By assuming for φ special functions we can obtain useful and easily applied criteria for proving that special equations of the form (1) are non-oscillatory. We add a few such criteria, noting to the left the function φ used, and to the right the special conditions, if any, which must be satisfied if the formula is to be applied. In these formulæ m and a denote constants to which we may assign any real values we please.

$$(a) \quad \varphi = 1, \qquad q \leqq 0.$$

$$(b) \quad \varphi = e^{mx}, \qquad q \leqq -mp - m^2.$$

$$(c) \quad \varphi = x^{-m}, \qquad q \leqq \frac{mxp - m(m+1)}{x^2} \qquad (0 < a).$$

$$(d) \quad \varphi = m - x, \qquad q \leqq \frac{p}{m - x} \qquad (b < m).$$

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* By letting $\phi = e^{-\int_c^x \lambda dx}$ we obtain the following theorem :

If a function λ exists which, together with its first derivative, is continuous throughout the interval $a \leqq x \leqq b$, and which satisfies the condition

$$(8') \qquad q \leqq \lambda' - \lambda^2 + \lambda p \qquad (a \leqq x \leqq b),$$

then (1) *is non-oscillating in this interval.*

Conversely, the theorem of the text follows from this one ; so that the two theorems are precisely equivalent to each other.

Picard (l. c., p. 102) deduces by another method a theorem identical with the one just stated, except that the inequality (8′) is replaced by

$$q \leqq \lambda' - (\lambda - p/2)^2.$$

This inequality is always satisfied when (8′) is satisfied, but the converse is not true. It is only when $p = 0$ that Picard's result is as general as ours.

$$(e) \quad \varphi = m^2 - x^2, \qquad q \leqq \frac{2(1+xp)}{m^2 - x^2} \quad (-|m| < a < b < |m|).$$

$$(f) \quad \varphi = \sin m(x-a), \quad q \leqq m^2 - mp \operatorname{ctn} m(x-a) \quad \left(a < a < b < a + \frac{\pi}{m}\right) \quad (m > 0).$$

$$(g)^* \quad \varphi = e^{-m\int p\,dx} \qquad q \leqq m(1-m)p^2 + mp' \text{ (provided } p \text{ has a continuous derivative).}$$

Formulæ of this sort might of course be multiplied indefinitely. To show how they are to be applied let us consider the simplest case of Bessel's equation

$$y'' + \frac{1}{x}y' + y = 0. \tag{9}$$

Since p is here discontinuous at the point $x = 0$, we can consider only intervals which do not include, or even reach up to this point. Since the equation is unchanged by replacing x by $-x$, it will be sufficient to consider intervals in which x is positive.

Formulæ (a), (e), (g) yield us no information whatever with regard to this equation. Formula (b) is most serviceable here if we let $m = -1$. It then shows us that (9) is non-oscillatory in the interval $a \leqq x \leqq \frac{1}{2}$, where a is any small positive quantity. If we let $m = 2$ in formulæ (d) and (e), they each show us that (9) is non-oscillatory in the interval $a \leqq x \leqq b$, when $0 < a < b < 2$. This is the best result these two formulæ can be made to yield if we wish to consider an interval starting from a point arbitrarily near the point $x = 0$. Of the seven formulæ written above, (f) gives the best result when applied to intervals of the sort just described, since when we let $m = \frac{1}{2}\sqrt{2}$, $a = -\frac{\pi}{2m}$, it shows us that (9) is non-oscillatory in the interval $a \leqq x \leqq b$ when

$$0 < a < b < \frac{\pi\sqrt{2}}{2} = 2.22.$$

This result, as we shall see in a moment, is nearly as good as any method could give us.

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* The special case $m = 1$ of this formula is noteworthy for the particularly simple result $(q \leqq p')$ which it yields; while the special case $m = \frac{1}{2}$ gives us the result we should have obtained by reducing (1) to the binomial form by a change of dependent variable only.

If on the other hand we wished to consider, for this same equation (9), intervals lying at a great distance from the origin, none of the formulæ above yield good results, the best being again (f), which when $m = 1$ shows us that (9) is non-oscillatory in any interval of length $\pi/2$. A much better result may be obtained by letting

$$\varphi = \frac{1}{\sqrt{x}} \sin m(x - a) \qquad \left(a < a < b < a + \frac{\pi}{m}\right).$$

This function leads us to the inequality

$$1 \leqq m^2 - \frac{3m - 2}{4x^2},$$

which just fails to be satisfied for large values of x when $m = 1$. We thus see that if l is any positive constant less than π, a positive constant M exists such that, in every interval of length l throughout which $x > M$, the equation (9) is non-oscillatory. That the function we have just used should give us an interval which is nearly twice as long as that given by formula (f) is the more remarkable because when x is large these two functions φ are, throughout an interval of length π, very nearly proportional to each other. If they were exactly proportional they would obviously lead to the same result.

We now leave these illustrative applications to equation (9).

Although we originally deduced condition (8) and its special cases as an extension, to the general equation (1), of the condition $q \leqq 0$ which we had established for the binomial equation (2), it turns out that some of these conditions give results for equation (2) which go beyond the result from which we started. Thus if we apply (f) to the special case $p = 0$ we get the important theorem

If throughout an interval of length less than l

$$q \leqq \frac{\pi^2}{l^2},$$

the equation (2) *is non-oscillatory in this interval.**

Up to this point we have obtained merely sufficient conditions that a differential equation should be non-oscillatory.

* Cf. the proof here given of this familiar theorem of Sturm with that given by Picard (l. c.).

A condition which turns out to be also necessary is obtained by taking as the function φ a solution of (1):

A necessary and sufficient condition that (1) *should be non-oscillatory in the interval* $a \leqq x \leqq b$ *is that it should have a solution which does not vanish in this interval.*

That this is a sufficient condition* is seen at once from the fact that either this solution or its negative satisfies conditions (7) and (8). To prove that it is also a necessary condition, assume that the interval $a \leqq x \leqq b$ is non-oscillatory, and consider the two solutions y_1 and y_2 of (1) which satisfy the conditions

$$y_1(a) = 0, \quad y_1'(a) > 0; \quad y_2(b) = 0, \quad y_2'(b) < 0.$$

Since neither of these solutions can vanish again in the interval, and since they are positive in the neighborhood of a and b respectively, they must be positive throughout the remainder of the interval. Accordingly $y_1 + y_2$ is a solution of (1) which is positive throughout the whole interval.

From the theorem just proved follows immediately this result:

A necessary and sufficient condition that the equation (1) *is non-oscillatory in the interval* $a \leqq x \leqq b$ *is that the solution of* (1) *which vanishes at* a (*or, if we prefer, at* b), *but is not identically zero, does not vanish again in the interval.*

This theorem gives us what is theoretically a perfect test; we have merely actually to compute the solution of (1) which vanishes at a, in the form of a series say, and to see whether or not this solution vanishes again in the interval. The difficulties involved in the computation may of course be so great in any special case as to make this method practically useless.

This last theorem if applied to (9) shows us that this equation is non-oscillatory in the interval $a \leqq x \leqq b$, where a is any small positive quantity and b is the smallest positive root of the Bessel function $J_0(x)$, viz. $b = 2.40\cdots$.†

The condition (8) can also be stated in the following form which again gives us a necessary as well as a sufficient condition:

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* This also follows at once from the well known theorem of Sturm: *Between two successive roots of a solution of* (1) *lies one and only one root of any linearly independent solution;* and conversely this theorem follows from the theorem of the text.

† This is actually the largest value that can be given to b, since every other solution of (9) has a root smaller than this. The proof of this fact is complicated by the presence of a singular point of (9) at $x = 0$. See, however, BULLETIN, March, 1897, p. 211.

If r is a real function of x which, throughout the interval $a \leqq x \leqq b$, is continuous, and satisfies the condition $r \leqq 0$, then a necessary and sufficient condition that (1) be non-oscillatory in this interval is that the non-homogeneous equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r \tag{10}$$

have a solution y satisfying the relation

$$y > 0 \quad (a \leqq x \leqq b).*$$

That this is really a necessary condition is seen from the fact that if (1) is non-oscillatory it has a solution positive throughout the interval, and by adding a sufficiently large positive multiple of this solution of (1) to an arbitrarily chosen solution of (10) we get the positive solution of (10) desired.

Another very important form into which condition (8) can be thrown is the following:

The function q_1 being continuous in the interval $a \leqq x \leqq b$ and satisfying the condition

$$q_1 \geqq q \qquad (a \leqq x \leqq b),$$

the equation (1) will be non-oscillatory in this interval if the equation

$$y'' + py' + q_1 y = 0 \tag{11}$$

is non-oscillatory there.

For if (11) is non-oscillatory it has a solution φ positive throughout the interval $a \leqq x \leqq b$. Substituting φ in the first member of (1) gives us, when we take account of the fact that φ satisfies (11),

$$\varphi'' + p\varphi' + q\varphi = (q - q_1)\varphi \leqq 0 \qquad (a \leqq x \leqq b).$$

Thus φ satisfies conditions (7) and (8), and therefore (1) is non-oscillatory.†

In conclusion I will mention that condition (7) may be replaced by the somewhat less restrictive condition

$$\varphi > 0 \qquad (a < x < b), \tag{7'}$$

* This theorem may also be proved directly and the other theorems of this paper deduced from it.

† This theorem may be proved directly by means of the methods used by Sturm (*Liouville's Journal*, vol. 1, p. 106) and is in fact a special case of one of Sturm's theorems. From it may be deduced the other theorems of this paper.

i. e., the case in which the function φ vanishes at one or both ends of the interval need not be excluded. The interval on the t-axis would, however, then extend to infinity in one or both directions, and the fundamental theorem concerning equation (2) from which we started would on longer be sufficient, but would have to be replaced by a theorem which states that, if $q \leqq 0$, no solution of (2) which vanishes at a finite point can approach a finite limit as x becomes either positively or negatively infinite, and that no solution of (2) can approach finite limits both when $x = +\infty$ and when $x = -\infty$.

The extension which our other theorems gain by the use of (7′) in place of (7) is easily seen. In using functions φ which vanish at one of the ends of the interval it is useful to know that if φ' also vanishes then φ cannot possibly satisfy (8),—a fact whose proof we also omit.

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CONCERNING REAL AND COMPLEX CONTINUOUS GROUPS.

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1. THIS paper aims to illustrate certain differences and certain analogies between related real and complex continuous groups. Lie's theory has been developed chiefly for the latter groups, the modifications necessary for real groups being treated quite briefly.

In §§ 2–4 are exhibited a real group in m variables and a real group in $2m$ variables, each of m^2 parameters, such that the corresponding complex groups are of like structure. In §§ 5–8, it is shown for $m = 2$ that the two real groups have different structures. Of the three proofs given, the first two are analytic and involve little technical knowledge of group theory, while the third group is geometric and gives a better insight into the nature of the question.

In § 10, it is illustrated for the case $m = 2$ how the general m-ary linear homogeneous complex continuous group gives rise to an isomorphic $2m$-ary linear homogeneous real continuous group. Similarly, the complex projective groups lead to groups of birational quadratic transformations.

The investigation has direct contact with the author's determination* of the structure of the largest group in the $GF[p^{2n}]$ leaving invariant $\xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 + \cdots + \xi_m\bar{\xi}_m$, where $\bar{\xi}_i$ is conjugate to ξ_i with respect to the $GF[p^n]$; also with the paper by Moore† on the universal invariant of finite groups of linear substitutions.

2. Consider the group G_m of all substitutions

$$S: \qquad \xi_i' = \sum_{j=1}^{m} a_{ij}\xi_j \qquad (i = 1, \cdots, m),$$

the coefficients and variables being complex numbers, such that S leaves formally invariant the Hermitian form

$$\Phi \equiv \xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 + \cdots + \xi_m\bar{\xi}_m.$$

The conditions upon the coefficients are seen to be

$$(1) \qquad \sum_{i=1}^{m} a_{ij}\bar{a}_{ij} = 1, \quad \sum_{i=1}^{m} a_{ij}\bar{a}_{ik} = 0 \qquad (j, k = 1, \cdots, m; j \neq k).$$

It follows that the inverse of S has the form

$$S^{-1}: \qquad \xi_i' = \sum_{j=1}^{m} \bar{a}_{ji}\xi_j \qquad (i = 1, \cdots, m).$$

The group G_m is evidently continuous. To obtain the general infinitesimal transformation, set $I^2 = -1$ and

$$a_{ii} = 1 + (a_{ii} + Ib_{ii})\,\delta t, \quad a_{ij} = (a_{ij} + Ib_{ij})\,\delta t$$

$$(i, j = 1, \cdots, m; j \neq i).$$

Substituting these values in the relations (1) and retaining only the first power of δt, we find that

$$1 + 2a_{jj}\delta t = 1, \quad (a_{jk} + a_{kj}) + I(b_{kj} - b_{jk}) = 0$$
$$(j, k = 1, \cdots, m; j \neq k).$$

The conditions upon the general infinitesimal transformation

$$(2) \qquad \delta\xi_i \equiv \xi_i' - \xi_i = \sum_{j=1}^{m} (a_{ij} + Ib_{ij})\,\delta t \cdot \xi_j$$

are therefore the following

* *Math. Annalen*, vol. 52, pp. 561–581.
† *Math. Annalen*, vol. 50, pp. 213–219.

$$a_{jk} = -a_{kj}, \quad b_{jk} = b_{kj} \qquad (j, k = 1, \cdots, m). \tag{3}$$

The general infinitesimal transformation of G_{m} is therefore a linear combination with *real* constant coefficients of

$$\begin{gathered} B_{ii}' \equiv I\xi_i \frac{\partial f}{\partial \xi_i}, \\ B_{ij}' \equiv I\xi_j \frac{\partial f}{\partial \xi_i} + I\xi_i \frac{\partial f}{\partial \xi_j}, \quad A_{ij}' \equiv \xi_j \frac{\partial f}{\partial \xi_i} - \xi_i \frac{\partial f}{\partial \xi_j}. \end{gathered} \tag{4}$$

Here B_{ij}' was obtained from (3) by setting $b_{ij} = b_{ji} = 1$ and the remaining constants all zero; A_{ij}' by setting $a_{ij} = -a_{ji} = 1$ and the remaining coefficients all zero.

The number of linearly independent transformations (4) is evidently m^2. If *complex* multipliers were allowed, we could derive from (4) the m^2 transformations

$$\xi_j \frac{\partial f}{\partial \xi_i} \qquad (i, j = 1, \cdots, m),$$

and therefore the general transformation of the m-ary linear homogeneous continuous group.

3. We obtain a continuous group R_{2m} on $2m$ real variables with real coefficients by replacing ξ_i by $X_i + IY_i$ for $i = 1, \cdots, m$ and separating reals and pure imaginaries. Relation (2) gives

$$\begin{aligned} X_i' + IY_i' - X_i - IY_i = \sum_{j=1}^{m} \{ &(a_{ij}X_j - b_{ij}Y_j) \\ &+ I(a_{ij}Y_j + b_{ij}X_j)\} \delta t. \end{aligned}$$

Hence the general infinitesimal transformation of R_{2m} is

$$\begin{gathered} \delta X_i = \sum_{j=1}^{m} (a_{ij}X_j - b_{ij}Y_j)\delta t, \quad \delta Y_i = \sum_{j=1}^{m} (b_{ij}X_j + a_{ij}Y_j) \\ (i = 1, \cdots, m). \end{gathered} \tag{5}$$

Denote by B_{ij} the transformation obtained by setting $b_{ij} = b_{ji} = 1$ and the remaining coefficients equal to zero; by A_{ij} that obtained by setting $a_{ij} = -a_{ji} = 1$ and the remaining coefficients equal to zero. Employing the usual abbreviations $p_i \equiv \dfrac{\partial f}{\partial X_i}$, $q_i \equiv \dfrac{\partial f}{\partial Y_i}$, we have

$$\begin{aligned} B_{ii} &\equiv X_i q_i - Y_i p_i, \quad B_{ij} \equiv X_j q_i - Y_j p_i + X_i q_j - Y_i p_j, \\ &\qquad A_{ij} \equiv Y_j q_i + X_j p_i - Y_i q_j - X_i p_j. \end{aligned}$$

Since $B_{ij} \equiv B_{ji}$, $A_{ij} \equiv -A_{ji}$, there are exactly m^2 independent

transformations, from which the general infinitesimal transformation of R_{2m} may be derived as a linear expression with real coefficients. In view of the identity

$$\Phi \equiv \sum_{i=1}^{m} \xi_i \bar{\xi}_i = \sum_{i=1}^{m} (X_i^2 + Y_i^2)$$

the group R_{2m} is an orthogonal group. As a check it may be verified that the transformations B_{ij}, A_{ij} leave Φ absolutely invariant.

4. The following commutator (Klammerausdruck) relations are readily formed:

$$(6)\quad \begin{matrix} (B_{ii}B_{ij})=0, & (B_{ii}B_{ij})=A_{ij}, & (B_{ii}A_{ij})=-B_{ij}, & (B_{ij}B_{ik})=A_{jk}, \\ (A_{ij}B_{ik})=B_{jk}, & (A_{ij}A_{ik})=A_{jk}, & (B_{ij}A_{ij})=2B_{ii}-2B_{jj}, & \end{matrix}$$

for $i, j, k = 1, \cdots, m$, with i, j, k distinct. If both subscripts of one symbol be different from the subscripts of the other, their commutator is zero.

It is readily verified that the transformations B_{ij}', A_{ij}' of §2 satisfy the commutator relations (6). This property would be expected to follow from the connection between G_m and R_{2m}. We may conclude that the continuous group with complex coefficients which is generated by the transformations B_{ij}, A_{ij} is isomorphic with the continuous complex group generated by B_{ij}', A_{ij}'.

Denote by B_{ij}'' the symbol obtained upon dropping the factor I from the symbol B_{ij}'. In the domain of real numbers, the transformations B_{ij}'', A_{ij}' generate the continuous group G_m' of all real linear homogeneous transformations in m variables. The symbols B_{ij}'', A_{ij}' do not satisfy the commutator relations (6). It is shown in §§ 5–8 that there does not exist in the real group G_2' ($m = 2$) a set of four independent infinitesimal transformations which satisfy the commutator relations (6), so that G_2' and R_4 are non-isomorphic real continuous groups of four parameters each.

5. For $m = 2$, the relations (6) are the following:

$$(B_{11}B_{22}) = 0, \quad (B_{11}B_{12}) = A_{12}, \quad (B_{11}A_{12}) = -B_{12},$$

$$(B_{22}B_{12}) = -A_{12}, \quad (B_{22}A_{12}) = B_{12}, \quad (B_{12}A_{12}) = 2B_{11} - 2B_{22}.$$

The first derived group is therefore the three-parameter group generated by A_{12}, B_{12}, $B_{11} - B_{22}$. The only (ausgezeichnete) transformation whose commutator with B_{11}, B_{22}, B_{12}, A_{12} is zero is seen to be $B_{11} + B_{22}$, aside from a constant

factor. Hence, if R_4 be isomorphic with G_2', $B_{11}+B_{22}$ must correspond with $\xi_1\frac{\partial f}{\partial \xi_1}+\xi_2\frac{\partial f}{\partial \xi_2}$ and the above three-parameter group with the first derived group of G_2'. To normalize R_4, set

$$Z_1\equiv\tfrac{1}{2}(B_{11}-B_{22}),\quad Z_2\equiv\tfrac{1}{2}A_{12},\quad Z_3\equiv-\tfrac{1}{2}B_{12},\quad Z_4\equiv B_{11}+B_{22}.$$

The above commutator relations then give

$$(7)\qquad (Z_1Z_2)=Z_3,\quad (Z_3Z_1)=Z_1,\quad (Z_2Z_3)=Z_2,$$

$$(8)\qquad (Z_4Z_1)=0,\quad (Z_4Z_2)=0,\quad (Z_4Z_3)=0.$$

The first derived group of G_2' is generated by

$$V_1\equiv\xi_1\frac{\partial f}{\partial \xi_2},\quad V_2=\xi_1\frac{\partial f}{\partial \xi_1}-\xi_2\frac{\partial f}{\partial \xi_2},\quad V_3\equiv\xi_2\frac{\partial f}{\partial \xi_1},$$

subject to the commutator relations

$$(9)\quad (V_1V_2)=-2V_1,\quad (V_2V_3)=-2V_3,\quad (V_3V_1)=-V_2.$$

To establish the non-isomorphism of R_4 and G_2', it suffices to prove that their first derived groups are non-isomorphic when considered as *real* continuous groups.

6. The most natural method of proof consists in showing that it is impossible to determine linear combinations of V_1, V_2, V_3 with real constant coefficients

$$Z_i'\equiv a_iV_1+b_iV_2+c_iV_3\qquad (i=1,2,3),$$

of determinant $\Delta\equiv\Sigma a_1b_2c_3\neq 0$, which satisfy relations (7). We observe that

$$(Z_1'Z_2')\equiv-2V_1(a_1b_2-b_1a_2)+V_2(a_1c_2-c_1a_2)-2V_3(b_1c_2-c_1b_2).$$

The conditions that the right member shall equal Z_3' are

$$(10)\quad a_1b_2-b_1a_2=-\tfrac{1}{2}a_3,\quad a_1c_2-c_1a_2=b_3,\quad b_1c_2-c_1b_2=-\tfrac{1}{2}c_3.$$

By advancing the subscripts of a_i, b_i, c_i, we obtain the conditions for the identities $(Z_2'Z_3')=Z_1'$, $(Z_3'Z_1')=Z_2'$

$$(11)\quad a_2b_3-b_2a_3=-\tfrac{1}{2}a_1,\ a_2c_3-c_2a_3=b_1,\ b_2c_3-c_2b_3=-\tfrac{1}{2}c_1,$$

$$(12)\quad a_3b_1-b_3a_1=-\tfrac{1}{2}a_2,\ a_3c_1-c_3a_1=b_2,\ b_3c_1-c_3b_1=-\tfrac{1}{2}c_2.$$

In view of the relations (11) and (12),

$$-\tfrac{1}{2}\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}=a_3\Delta.$$

Applying the first relation (10), $\frac{1}{4}a_3=a_3\Delta$. In a similar manner, or by advancing the subscripts (which does not alter Δ), we find

$$\tfrac{1}{4}a_1 = a_1\Delta, \quad \tfrac{1}{4}a_2 = a_2\Delta.$$

Since a_1, a_2, a_3 are not all zero, it follows that $\Delta = \frac{1}{4}$.

Multiplying equations (12) by c_2, $-b_2$, a_2 respectively and adding the resulting equations, we find

$$\Delta = -b_2^2 - a_2c_2. \tag{13}$$

Employing the multipliers c_1, $-b_1$, a_1, we find

$$0 = -b_1b_2 - \tfrac{1}{2}a_2c_1 - \tfrac{1}{2}a_1c_2. \tag{14}$$

In a similar manner, or by advancing the subscripts,

$$\Delta = -b_1^2 - a_1c_1. \tag{15}$$

By (14) and the second equation of set (10),

$$b_3^2 = (a_1c_2 - c_1a_2)^2 = 4b_1^2b_2^2 - 4a_1c_1a_2c_2.$$

Eliminating a_1c_1 and a_2c_2 by (15) and (13), and setting $\Delta = \frac{1}{4}$,

$$b_1^2 + b_2^2 + b_3^2 = -\tfrac{1}{4}.$$

But this equation is impossible for real values of the b_i.

7. A second proof is derived from the following investigation which gives certain interesting properties of the group I' generated by Z_1, Z_2, Z_3, subject to the relations (7). Set

$$\left.\begin{aligned} Z_1 &= \alpha_1U_1 + \alpha_2U_2 + \alpha_3U_3, \\ Z_2 &= \beta_1U_1 + \beta_2U_2 + \beta_3U_3, \\ Z_3 &= \gamma_1U_1 + \gamma_2U_2 + \gamma_3U_3, \end{aligned}\right\} \qquad \Delta \equiv \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \neq 0.$$

We obtain by solution the most general set of independent infinitesimal transformations U_1, U_2, U_3, of the group I'. We seek the commutator relations of the U_i. Denote by α_i' the first minor (without prefixed sign) of α_i in Δ, β_i' the first minor of β_i, γ_i' the first minor of γ_i. Form (Z_2Z_3) and equate the result to Z_1; expand similarly $(Z_3Z_1) = Z_2$, $(Z_1Z_2) = Z_3$. The results are

$$\left.\begin{aligned} Z_1 &= \alpha_1'(U_2U_3) - \alpha_2'(U_3U_1) + \alpha_3'(U_1U_2), \\ Z_2 &= -\beta_1'(U_2U_3) + \beta_2'(U_3U_1) - \beta_3'(U_1U_2), \\ Z_3 &= \gamma_1'(U_2U_3) - \gamma_2'(U_3U_1) + \gamma_3'(U_1U_2). \end{aligned}\right.$$

The determinant of the coefficients equals Δ^2, being equal to the determinant of the first minors of Δ. Moreover,

$$\begin{vmatrix} \beta_2' & \beta_3' \\ \gamma_2' & \gamma_3' \end{vmatrix} = \alpha_1\Delta.$$

The solution of the above relations therefore gives

$$\Delta(U_2U_3) = a_1Z_1 + \beta_1Z_2 + \gamma_1Z_3,$$
$$\Delta(U_3U_1) = a_2Z_1 + \beta_2Z_2 + \gamma_2Z_3,$$
$$\Delta(U_1U_2) = a_3Z_1 + \beta_3Z_2 + \gamma_3Z_3.$$

The matrix of the coefficients on the right is the transposed of the matrix of the coefficients of the U_i in the expressions for the Z_i. Eliminating the Z_i, we have

	U_1	U_2	U_3
$\Delta(U_2U_3)=$	$a_1^2+\beta_1^2+\gamma_1^2$	$a_1a_2+\beta_1\beta_2+\gamma_1\gamma_2$	$a_1a_3+\beta_1\beta_3+\gamma_1\gamma_3$
$\Delta(U_3U_1)=$	$a_1a_2+\beta_1\beta_2+\gamma_1\gamma_2$	$a_2^2+\beta_2^2+\gamma_2^2$	$a_2a_3+\beta_2\beta_3+\gamma_2\gamma_3$
$\Delta(U_1U_2)=$	$a_1a_3+\beta_1\beta+\gamma_1\gamma_3$	$a_2a_3+\beta_2\beta_3+\gamma_2\gamma_3$	$a_3^2+\beta_3^2+\gamma_3^2$

The symmetry of the matrix of coefficients is in accord with a known property of the group.*

In order that the transformations U_i should satisfy the same commutator relations (9) as the transformations V_i it is necessary that $(U_2U_3) = -2U_3$, so that $a_1^2 + \beta_1^2 + \gamma_1^2 = 0$. For real values of a_1, β_1, γ_1, this requires $a_1 = \beta_1 = \gamma_1 = 0$, contrary to hypothesis. Hence the real group Γ' of the Z_i is not isomorphic with the real group of the V_i.

To obtain the most general set of three infinitesimal transformations U_i of Γ which satisfy the same commutator relations (7) as the transformations Z_i themselves,

$$(U_2U_3) = U_1, \quad (U_3U_1) = U_2, \quad (U_1U_2) = U_3,$$

it is necessary and sufficient to take solutions a_i, β_i, γ_i of

$$1 = a_1^2 + \beta_1^2 + \gamma_1^2, \quad 1 = a_2^2 + \beta_2^2 + \gamma_2^2, \quad 1 = a_3^2 + \beta_3^2 + \gamma_3^2.$$
$$0 = a_1a_2 + \beta_1\beta_2 + \gamma_1\gamma_2, \quad 0 = a_1a_3 + \beta_1\beta_3 + \gamma_1\gamma_3,$$
$$0 = a_2a_3 + \beta_2\beta_3 + \gamma_2\gamma_3.$$

These are the conditions for an orthogonal substitution, the invariant relation being

$$Z_1^2 + Z_2^2 + Z_3^2 \equiv \Delta\,(U_1^2 + U_2^2 + U_3^2).$$

8. To give a third proof, based upon geometric considerations, it suffices to consider the adjoint groups of the three-parameter groups in question. The adjoint of the group of the V_i is

$$2e_2\frac{\partial f}{\partial e_1} - e_3\frac{\partial f}{\partial e_2}, \quad -2e_1\frac{\partial f}{\partial e_1} + 2e_3\frac{\partial f}{\partial e_3}, \quad e_1\frac{\partial f}{\partial e_2} - 2e_2\frac{\partial f}{\partial e_3},$$

having as its only invariant curve the real conic

*Lie-Scheffers, Vorlesungen über continuierliche Gruppen, p. 567.

$$e_1 e_3 + e_2^2 = 0. \tag{16}$$

The adjoint group of the group of the Z_i is

$$e_3\frac{\partial f}{\partial e_2} - e_2\frac{\partial f}{\partial e_3}, \quad -e_3\frac{\partial f}{\partial e_1} + e_1\frac{\partial f}{\partial e_3}, \quad e_2\frac{\partial f}{\partial e_1} - e_1\frac{\partial f}{\partial e_2},$$

having as its only invariant the imaginary conic

$$e_1^2 + e_2^2 + e_3^2 = 0. \tag{17}$$

Now the replacement of one complete set of independent infinitesimal transformations of a group by a second complete set merely gives rise to a linear homogeneous transformation upon the variables e_i of the adjoint group. The latter will be a real transformation if the second set is expressed in terms of the first by real coefficients. Since the equations (16) and (17) can not be transformed into each other by a real ternary substitution, it follows that the transformations V_i are not expressible as real linear functions of the Z_i.

The method of reduction of three-parameter non-integrable groups to a normal type given in Lie-Scheffers, Vorlesungen, pp. 566–568 is immediately applicable only to complex groups. For real groups there are two (and, indeed, only two*) distinct cases, according as the invariant conic (necessarily non-degenerate) is real or imaginary. The two methods there given as optional for complex groups are to be differentiated for real groups to correspond to the cases of real and imaginary conics, yielding respectively the normal types (I) and (I′) of p. 568, or types (9) and (7) respectively of this paper.

9. The real four-parameter groups G_4' and R_4 have been proved to have different structures. Applying the imaginary transformation of variables

$$X_1 = x_1, \quad Y_1 = y_1, \quad X_2 = Ix_2, \quad Y_2 = Iy_2 \qquad (I^2 = -1),$$

the infinitesimal transformations B_{11}, B_{22}, B_{12}, A_{12} of R_4 become

$$b_{11} \equiv x_1\frac{\partial f}{\partial y_1} - y_1\frac{\partial f}{\partial x_1}, \quad b_{22} \equiv x_2\frac{\partial f}{\partial y_2} - y_2\frac{\partial f}{\partial x_2},$$

$$b_{12} \equiv x_2\frac{\partial f}{\partial y_1} - y_2\frac{\partial f}{\partial x_1} - x_1\frac{\partial f}{\partial y_2} + y_1\frac{\partial f}{\partial x_2},$$

$$a_{12} \equiv y_2\frac{\partial f}{\partial y_1} + x_2\frac{\partial f}{\partial x_1} + y_1\frac{\partial f}{\partial y_2} + x_1\frac{\partial f}{\partial x_2}.$$

* An irreducible ternary quadratic form with real coefficients is reducible either to $b(e_1^2 + e_2^2 + e_3^2)$ or to $b(e_1^2 + e_2^2 - e_3^2)$.

They satisfy the commutator relations

$$(b_{11}b_{22}) = 0, \quad (b_{11}b_{12}) = a_{12}, \ (b_{11}a_{12}) = -b_{12},$$
$$(b_{22}b_{12}) = -a_{12}, \ (b_{22}a_{12}) = b_{12}, \ (b_{12}a_{12}) = -2b_{11} + 2b_{22}.$$

Except for the last relation, these are identical with the commutator relations of B_{11}, B_{12}, B_{22}, A_{12} (§5). Setting

$$W_1 \equiv a_{12}, \ W_2 \equiv -b_{12}, \ W_3 \equiv b_{11} - b_{22}, \ W_4 \equiv b_{11} + b_{22},$$

we have the commutator relations

$$(W_1W_2) = -2W_3, \ (W_2W_3) = 2W_1, \ (W_3W_1) = 2W_2,$$
$$(W_4W_1) = 0, \quad (W_4W_2) = 0, \quad (W_4W_3) = 0.$$

These relations are also satisfied by the transformations

$$w_1 \equiv xq + yp, \ w_2 \equiv xp - yq, \ w_3 \equiv xq - yp, \ w_4 \equiv xp + yq,$$

which generate the general binary group G_2'. *By an imaginary transformation of variables, R_4 may be given a real form having the same structure as G_2'.*

10. Consider the group G of all binary transformations

$$S: \quad \begin{cases} X' = \alpha X + \gamma Y \\ Y' = \beta X + \delta Y \end{cases} \qquad (\alpha\delta - \beta\gamma = 1)$$

upon complex variables X, Y with complex coefficients of determinant unity. Let $I^2 = -1$ and set

$$X = x + Ix_1, \ Y = y + Iy_1, \ \alpha = a + Ia_1, \ \beta = b + Ib_1, \text{ etc.}$$

Then S corresponds to the quaternary transformation

$$\Sigma: \quad \begin{array}{c|cccc} & x & x_1 & y & y_1 \\ \hline x' = & a & -a_1 & c & -c_1 \\ x_1' = & a_1 & a & c_1 & c \\ y' = & b & -b_1 & d & -d_1 \\ y_1' = & b_1 & b & d_1 & d \end{array}$$

The relation $\alpha\delta - \beta\gamma = 1$ gives

$$ad - bc - a_1d_1 + b_1c_1 = 1, \quad ad_1 + a_1d - bc_1 - b_1c = 0.$$

The determinant of Σ is seen to equal

$$(ad - bc - a_1d_1 + b_1c_1)^2 + (ad_1 + a_1d - bc_1 - b_1c)^2 = 1.$$

To the product S_1S_2 of two substitutions of the form S, corresponds the product $\Sigma_1\Sigma_2$ of the corresponding substitutions Σ. Hence the group G is isomorphic with the

group Γ of the substitution Σ. But Σ reduces to the identity only when S is the identity. Hence the isomorphism is holoedric.

By the usual method, the general infinitesimal transformation of Γ is found to be a linear combination of the following linearly independent transformations:

	$\frac{\partial f}{\partial x}$	$\frac{\partial f}{\partial x_1}$	$\frac{\partial f}{\partial y}$	$\frac{\partial f}{\partial y_1}$
A	x	x_1	$-y$	$-y_1$
A_1	$-x_1$	x	y_1	$-y$
B	0	0	x	x_1
B_1	0	0	$-x_1$	x
C	y	y_1	0	0
C_1	$-y_1$	y	0	0

The real group Γ therefore possesses no invariant. The non-vanishing second minors of the matrix of coefficients are

$$P \equiv x^2 + x_1^2, \quad Q \equiv y^2 + y_1^2, \quad R = xy + x_1y_1, \quad S = x_1y - xy_1.$$

Upon them the group Γ gives rise to the following transformations:

	$\frac{\partial f}{\partial P}$	$\frac{\partial f}{\partial Q}$	$\frac{\partial f}{\partial R}$	$\frac{\partial f}{\partial S}$
A	$2P$	$-2Q$	0	0
A_1	0	0	$-2S$	$2R$
B	0	$2R$	P	0
B_1	0	$-2S$	0	$-P$
C	$2R$	0	Q	0
C_1	$2S$	0	0	Q

The determinants of the fourth order of this matrix are all identically zero. To obtain the homogeneous invariants, we annex Euler's homogeneous operator

$$H \equiv P\frac{\partial f}{\partial P} + Q\frac{\partial f}{\partial Q} + R\frac{\partial f}{\partial R} + S\frac{\partial f}{\partial S}.$$

The determinant of the coefficients of A, A_1, B, H equals $8PR(S^2+R^2-PQ)$. The determinant of A, A_1, B_1, H equals $-8PS(S^2+R^2-PQ)$. In this way the only homogeneous invariant is seen to be

$$\psi \equiv S^2 + R^2 - PQ.$$

In terms of the initial variables x, x_1, y, y_1, we see that ψ vanishes identically. Also

$$P \equiv |x + Ix_1| \equiv |X|, \quad Q \equiv |Y|,$$

$$\frac{Y}{X} \equiv \frac{R - IS}{P}, \quad \frac{X}{Y} \equiv \frac{R + IS}{Q}.$$

The group G is hemiedrically isomorphic with the group of linear fractional substitutions

$$Z' = \frac{\alpha + \gamma Z}{\beta + \delta Z}, \quad Z \equiv \frac{Y}{X}. \tag{18}$$

The quaternary group on P, Q, R, S is isomorphic with a ternary fractional group on $Q/P, R/P, S/P$. But

$$\frac{Q}{P} \equiv \left(\frac{R}{P}\right)^2 + \left(\frac{S}{P}\right)^2.$$

Eliminating Q/P, we obtain a group of birational quadratic transformations in the plane. It may evidently be obtained more directly from the transformations (18).

THE UNIVERSITY OF CHICAGO,
January, 1901.

ON HOLOMORPHISMS AND PRIMITIVE ROOTS.

BY DR. G. A. MILLER.

(Read before the American Mathematical Society, February 23, 1901.)

IN an earlier note* it was observed that every holomorphism of an abelian group with itself can be obtained by establishing an isomorphism between the abelian group and one of its subgroups (which may sometimes be the entire group) and associating the product of corresponding operators with the original operator of the group. The present note is devoted to some additional developments along this line and especially to some elementary results in the theory of numbers which may be derived by this method.

Let s_1 represent an operator of order p^m (p being any prime number) and let P, the group generated by s_1, be

* BULLETIN, Vol. 6 (1900), p. 337.

made isomorphic with one of its subgroups of order p^{m_1}, $m_1 < m$. Each operator of P is transformed, by some operator t in the group of isomorphisms of P, into itself multiplied by the corresponding operator in this isomorphism. Assuming that $t^{-1} s_a t = s_{a+1}{}^{s_a}$, we have

$$t^{-n} s_a t^n = s_{a+n}{}^{s_a^n} s_{a+n-1}{}^{\frac{n(n-1)\cdots(n-r+1)}{r!}} \cdots s_{a+1}{}^{s_a}.$$

It is easy to prove that the order of the product of the operators which are multiplied into s_a is equal to the order of s_{a+1}^n whenever p is odd. In case n is prime to p this follows directly from the fact that each of the factors which precede s_{a+1} is of a lower order than s_{a+1}. In general, let $n = kp_\lambda$, k being prime to p. The exponent of $s_{a+\beta}$ is divisible by $p^{\lambda - m'}$, where m' is the exponent of the highest power of p that is contained in β, since the product of n successive numbers is divisible by $n!$. As the order of $s_{a+\beta}$ does not exceed the order of $(s_{a+1})^{p^{\beta-1}}$, the order of the power of $s_{a+\beta}$ which occurs in the above formula cannot exceed $(s_{a+1})^{p^{\lambda+\beta-m'-1}}$. Hence it is less than the order of s_{a+1}^n whenever $\beta > 1$, and *the product of all the factors which are multiplied into s_a is of the same order as s_{a+1}^n when p is odd.*

When p is even we assume that $m_1 < m - 1$. With the same notation as above it is clear that the order of $s_{a+\beta}$ does not exceed the order of $(s_{a+1})^{2^{2(\beta-1)}}$. Hence the order of the power of $s_{a+\beta}$ in the formula cannot exceed

$$(s_{a+1})^{2^{\lambda+2\beta-m'-2}}$$

As $m' + 2$ is less than 2β whenever $\beta > 1$, it follows that in this case the order of the product of the factors which are multiplied into s_a is again equal to s_{a+1}^n. Hence *t is always of order p^{m_1} and the group of isomorphisms of P contains a cyclic subgroup of order p^{m-1} when p is odd and one of order 2^{m-2} when p is even.*

The group of isomorphisms of a cyclic group is abelian,* and can be represented as a regular substitution group whose elements correspond to the operators of highest order in the cyclic group.† Hence the group of isomorphisms of the cyclic group of order p^m is of order $p^{m-1}(p-1)$. In particular, the group of isomorphisms I of the cyclic group of order 2^m is of order 2^{m-1}. We have just found that I contains a cyclic subgroup of order 2^{m-l}, formed by all its operators which transform into itself an operator of order 2^l $(l > 1)$ in P.

* *Trans. Amer. Math. Soc.*, vol. 1 (1900), p. 397.

† The order of a cyclic group is said to have primitive roots whenever its group of isomorphisms is cyclic.

The group I contains an operator s' of order two which transforms each operator of P into its inverse. As s' is not contained in the above cyclic subgroup of order 2^{m-2}, which is composed of all the operators of I which transform an operator of order four in P into itself, it and this subgroup must generate I.

It is now easy to determine the exponent to which a number belongs mod 2^m, since this exponent is the order of the corresponding operator in I. In the above mentioned cyclic subgroup of order 2^{m-2}, an operator of order 2^k is commutative with the operators of order 2^{m-k} in P, but not with those of order 2^{m-k+1}. From this fact and the fact that s' transforms each operator of P into its inverse it follows that all the numbers which belong to exponent 2^k $(k>1)$ mod 2^m are of the form $\pm (l\,2^{m-k}+1)$ where l is any one of the $\varphi(2^k)$ numbers not greater than 2^k and prime to 2^k; and *vice versa*. When $k=1$ we have to add 2^m-1 to the numbers obtained in this way. Hence the numbers which belong to exponent 2^{m-2} are $\equiv 3$ or 5 mod 8.*

From what is proved above it follows that the group of isomorphisms I_1 of P contains a cyclic subgroup of order p^{m-1}, p being any odd prime, which is composed of all the operators of I_1 commutative with each of the operators of order p in P. By adding to this subgroup the operators which transform transitively these $p-1$ operators of order p, we obtain the $p^{m-1}(p-1)$ operators of I. The group of order $p-1$ according to which the operators of order p are transformed contains no more than d operators whose orders divide d, any factor of $p-1$, since $x^d \equiv 1$ mod p can have no more than d roots.† Hence it cannot have two subgroups of the same order and must therefore be cyclic. Since I_1 is abelian its operators of highest order are obtained by multiplying the operators of order $p-1$ in this cyclic subgroup by the operators of order p^{m-1} in the above mentioned cyclic subgroup of order p^{m-1}. Hence I_1 is cyclic and the primitive roots of p^m are also primitive roots of p.‡ It may be observed that *the above furnishes an independent proof of the existence of primitive roots of* p^m. That the cyclic group of order $2p^m$ has the same group of isomorphisms as the cyclic group of order p^m follows directly from the fact that the operator of order two in the former must correspond to itself in every holomorphism of the group with itself. Hence $2p^m$ also has primitive roots.

* Cf. Mathews, Theory of numbers, 1892, p. 30.

† Gauss, Disquisitiones Arithmeticæ, 1801, Art. 54.

‡ Ibid, Art. 92.

The holomorphisms mentioned in the second and third paragraphs show that all numbers of the form $kp^{m-a}+1$, where k is any one of the $\varphi(p^a)$ natural numbers which are not greater than p^a and prime to p^a, belong to the exponent p^a; a being any positive integer less than m when p is odd and less than $m-1$ when p is even. In the preceding paragraph it is proved that these are the only numbers which belong to an exponent which is a power of an odd prime. The product of all the numbers which belong to exponent p^{m-1}, and the $\varphi(p-1)$ powers of a number which is not a primitive root of p^m, but belongs to exponent $p-1 \bmod p$, will clearly give all the primitive roots of p^m, since the corresponding operators in I_1 are all its operators of order $p^{m-1}(p-1)$. In particular the primitive roots of 3^m are the products of 3^m-1 and the numbers of the form $3l+1$, l being any one of the positive integers not greater than 3^{m-1} and prime to 3^{m-1}.

The necessary and sufficient condition that an operator of I_1 corresponds to a primitive root of p^a is that its order is divisible by $p-1$ and that its $(p-1)$th power corresponds to a holomorphism of P with itself which may be obtained by establishing a p, 1 isomorphism between P and its subgroup of order p^{m-1}. Hence *the primitive roots of* p^a $(a>1)$ *are also the primitive roots of every power of* p.* The pth power of a primitive root of p is also a primitive root of p, but the pth power of a primitive root of p^a is not a primitive root of p^a. The primitive roots of p are therefore not always primitive roots of p^a. In fact, we observe directly, from the orders of the operators of I_1, that just $\dfrac{p-1}{p}$ of the primitive roots of p which are less than p^m are also primitive roots of p^m.

The preceding considerations can readily be applied to the general cyclic group C of order $2^{a_0}p_1^{a_1}p_2^{a_2}\cdots p_m^{a_m}$ ($p_1, p_2, \cdots, p_m$ being odd prime numbers). By making C isomorphic with its subgroup of order $2^{a_0'}p_1^{a_1'}p_2^{a_2'}\cdots p_m^{a_m'}$ (where $a_0'<a_0-1$; $a_\gamma'<a_\gamma$, $\gamma=1, 2, \cdots, m$), and multiplying the corresponding operators, we obtain a holomorphism of C with itself, which corresponds to an operator of order $2^{a_0'}p_1^{a_1'}p_2^{a_2'}\cdots p^{a_m'}$ in its group of isomorphisms. Since the latter group is the direct product† of the groups of isomorphisms of the cyclic groups of orders $2^{a_1}, p_1^{a_1}, p_2^{a_2}, \cdots, p_m^{a_m}$ and since the group of isomorphisms of each one of these groups involves operators of order two whenever the order of the group exceeds two, the

* Lebesgue, *Liouville's Journal*, vol. 19 (1854), p. 344.
† *Trans. Amer. Math. Soc.*, vol. 1 (1900), p. 396.

group of isomorphisms of C is cyclic only when $a_0 = 0$ or 1 and just one of the other exponents differs from 0, or when $a_0 = 1$ or 2 and all the other exponents are 0.*

CORNELL UNIVERSITY,
February, 1901.

BESSEL FUNCTIONS.

Einleitung in die Theorie der Bessel'schen Funktionen. By PROFESSOR J. H. GRAF und DR. E. GUBLER. Zweites Heft: *Funktionen zweiter Art.* Bern, Wyss and Co., 1900.

THE first part of this work appeared in 1898 and was reviewed in the BULLETIN, February, 1899, pp. 253–8. The general arrangement of the second part is similar to that of the first, the authors again emphasizing the fact that the work is done in the spirit of Schläfli's lectures, the manuscripts of which were in their hands, though many problems are extended and modernized. This fact explains the absence of many important phases of the theory of the Bessel functions which one might expect in a symmetric treatise. Moreover, the authors have been rather overgenerous in their references to papers originating at Bern, omitting others which contained proofs of fundamental theorems prior to their discovery by the Bern school, although probably no plagiarism could be charged. Several fundamental theorems by American authors have received no recognition in the book.

Here, as in Volume I, the loop integral is the principal factor in the investigation, and next in importance is the expansion in series. The differential equation is less frequently used. The procedure is rather original, and frequently markedly different proofs for well-known theorems are given, which in some instances have led to detection of error in papers already published.

The only attempt at a concrete illustration or application is the expansion of a few functions in terms of Bessel functions, though the relations which exist between these functions and others are quite fully brought out.

The second part begins with the expansion of $\frac{1}{x-y}$ in terms of Bessel functions, the result being

* Gauss, Disquisitiones Arithmeticæ, 1801, Art. 92.

$$\frac{1}{x-y} = J_0(y)\cdot\frac{1}{x} + \sum_{n=1}^{\infty} J_n(y)\int_0^{\frac{N}{x}} e^{-ns}\left(t^n - \frac{(-1)^n}{t^n}\right) ds.*$$

$$[s = \tfrac{1}{2}(t - t^{-1})], \quad (|x| > |y|), \quad (\lim N = \infty).$$

The part under the integral sign is denoted by $2O_n(x)$, and $O_n(x)$ is called the Bessel function of the second kind. This terminology is unusual, since the differential equation for $J_n(x)$ is not satisfied by $O_n(x)$, but in other respects $O_n(x)$ is quite analogous to $J_n(x)$. The authors suggest the analogy between the Bessel functions of the first and second kind on the one hand and spherical harmonics of the first and second kind, as defined by Neumann, on the other.

The symbol n is used to denote an integral parameter; the form of the infinite series for $O_n(x)$ is then derived, and the numerical coefficients calculated for $n=1$ to $n=11$. This method is then compared with that of Neumann for obtaining equation (1). $O_n(x)$, $J_n(x)$ are both shown to exist in a Laurent ring. Any continuous and differentiable function can be expanded in but one way in terms of Bessel functions.† The discussion of integrals of products of J, O closes the chapter.

In the following chapter the related function $S_n(x)$ is introduced:

$$S_n(x) = \int_1^{\frac{N}{x}} e^{-ns}\left(t^n - \frac{(-1)^n}{t^n}\right)\frac{dt}{t},$$

$$\left(O_n(x) = \frac{\cos^2 \tfrac{1}{2}n\pi}{x} + \frac{n}{2x} S_n(x)\right).$$

The expression $\cos^2 \frac{1}{2}n\pi$, $= 0$ or 1, $n = 1 \bmod 2$, $0 \bmod 2$ causes some confusion, both in this and later chapters, but the difficulty is easily removed by changing a limit in a summation. The numerical calculation of $S_n(x)$ is given for $n=1$ to $n=12$. $S_n(x)$ is always a polynomial in x^{-1}.

$$\frac{\cos^2 \tfrac{1}{2}\pi n}{x}$$

is eliminated and $S_n(x)$ expressed as a (finite) series in terms of $O_\lambda(x)$. The differential equation is found to be

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* The notation $\overset{a}{J}$ of the work reviewed is here replaced by J_a, etc.

† In my review of Part 1, I carelessly attributed this theorem to Schlömilch. This was simply an error of my own; the statement was not made in the book, but Schlömilch's name was used in a different connection.

$$x^2\frac{d^2}{dx^2}S_n + x\frac{dS_n}{dx} + (x^2 - n^2)S_n = 2x\sin^2\tfrac{1}{2}n\pi$$

$$+ 2n\cos^2\tfrac{1}{2}n\pi = 2x \text{ or } 2n \text{ as } n \equiv 0,\ 1 \bmod 2\ ;$$

that for O is

$$\square\, O_n + O_n = x\cos^2\tfrac{1}{2}n\pi + n\sin^2\tfrac{1}{2}n\pi,$$

$\square\, J_n(x) = 0$ being the differential equation for $J_n(x)$.

The chapter closes with the integration of this differential equation, which results in expressing $S_n(x)$ and $O_n(x)$ in terms of $J_n(x)$, $K_n(x)$.

Chapter VIII introduces two new functions, $T_n(x)$, $U_n(x)$, defined by series, somewhat analogous to the partial summations of $J(x)$ and of $K(x)$. They are next expressed as integrals,

$$T_n(x) = \frac{2}{\pi i}\int_0^\pi \left(\varphi - \frac{\pi}{2}\right) e^{-i(x\sin\phi - n\phi)}\, d\varphi.$$

In deriving the differential equation for $T_n(x)$, $\cos^2\frac{1}{2}n\pi$ intrudes again and the authors have chosen an infelicitous expression for removing it. In fact, taking the statement pp. 50–52 literally, an actual error would be made. This is about the only difficult part to follow in the text.

The function T is expressed in terms of J_λ and also as a definite integral—a similar discussion follows for U. K, J, T, S are shown to satisfy the relation

$$(-1)^n y_{-n} = y_n.$$

At the end of the chapter the relation between these functions and the y of Neumann, of Hankel, and of Weber are given, the first being expressed by the equation

$$y_n(x) = \log x.J_n(x) - \tfrac{1}{2}S_n(x) + \tfrac{1}{2}T_n(x) - U_n(x),$$

and the others are also given in detail. The functions S_n, T_n, U_n, are called Schläfli functions.

Chapter X deals with the addition theorem ; it is prefaced by a historical introduction which concisely gives the development of the problem. The method of proof is quite consequent : it consists in expressing $J_n(x + y)$ as a definite (loop) integral

$$J_n(x + y) = \frac{1}{2\pi i}\int e^{(x+y)s}t^{-n-1}dt, \qquad [s = \tfrac{1}{2}(t - t^{-1})],$$

$$e^{ys} = \sum_{\lambda=-\infty}^{\infty} J_\lambda(y)t^\lambda,$$

$$J_n(x+y) = \sum_{\lambda=-\infty} J_\lambda(y) J_{n-\lambda}(x), \qquad (|y|<|x|)$$

By simple and natural transformations, the functions

$$J_n(x), \quad K_n(x), \quad O_n(x), \quad S_n(x), \quad T_n(x),$$

are shown to satisfy the same addition theorem, namely,

$$Z_n(x+y) = \sum_{\lambda=-\infty}^{\infty} Z_{n-\lambda}(x) J_\lambda(y);$$

similar expressions are derived for the argument $x-y$, for parameter n and $-n$. The chapter concludes with a similar discussion of the product of two Bessel functions. To me this chapter appears very successful in its consistent method of procedure.

The next chapter deals with the expansion into a continued fraction of the ratio of two Bessel functions whose parameters differ by unity; it is preceded by a sketch of the historical development of the problem, a feature that is repeated in every subsequent chapter. Several pages are devoted to the expansion of particular functions.

The Schläfli function $P_m^{(a)}(x)$ is shown to be a polynomial of order m in x^{-1}. It is developed for values of m from -2 to 8 in terms of an arbitrary parameter a; then a recurring process gives the value of the function for other negative values of the integer m. The chapter closes with the application to some particular cases, $a=\frac{1}{2}$, $a=-\frac{1}{2}$, $\lim_{m=\infty} P_m^{(a)}(x)$.

Chapter XII treats of the classic problem of the relation of the Bessel function to the hypergeometric series. The treatment is quite original and direct, the method being somewhat independent of the older memoirs. The problem is to determine the value of the integral

$$S = \int_0^\infty J_a(x) e^{-bx} x^{c-1} dx;$$

the result is

$$S = \frac{\Gamma(a+c)}{2^a \Gamma(a+1) b^{a+c}} F\left(\frac{a+c}{2}, \frac{a+c+1}{2}, a+1, -\frac{1}{b^2}\right);$$

the conditions for convergence are carefully discussed.

An interesting case is discussed wherein $b=\pm i$; the region in which the function exists and the transformed path of the loop integral are well treated. A second proof is given, depending on a curvilinear integral; some new relations between F functions are incidentally found, among them being

$$F\left(\frac{a+\gamma-1}{2},\ \frac{a+\gamma}{2},\ \gamma,\ -\frac{4x(1-x)}{(1-2x)^2}\right)$$
$$= (1-2x)^{a+\gamma-1}(1-x)^{1-\gamma}F(a,\ 1-a,\ \gamma,\ x).$$

This last equation is then developed directly by means of the definite integral; the path of integration is varied and the moduli of periodicity obtained by crossing the section are then discussed. Here again the special power of the curvilinear integral is exhibited, in the use of which the authors have shown considerable skill. The function considered is

$$T = \frac{t^2}{4(t-1)},$$

and the curve of section (Grenzscheide) is a limaçon. The chapter closes with a discussion of a few particular cases, including besides some well known results a few new ones regarding integrals of Bessel functions.

In the final chapter Weber's discontinuous integral

$$\int_0^\infty J_1(x)J_0(ax)dx$$

is discussed and generalized for any parameters; this is done by Dr. Gubler. The method employed is an application of the principles established in the preceding chapter, with the selection of an appropriate path of integration for each case.

A short appendix is added, in which still another form of the integral is obtained, and another section reduces the differential equation of the Bessel function of the first kind to a Riccati equation; and gives the form of the solution as a function of two independent $J(x)$ functions.

The bibliography which introduces each chapter, and the list of sources quoted which is appended to each part, while not always complete and not always giving the original source, still form a valuable part of the book. This work and the treatise of Gray and Mathews supplement each other, each being somewhat one sided when studied alone.

A number of typographical errors already found are corrected in a list inserted in Part II; but a good many more are still in the text, though few, if any, would prove a source of annoyance to the reader.

Any worker in Bessel functions will find the work a helpful text.

VIRGIL SNYDER.

CORNELL UNIVERSITY,
March 9, 1901.

SHORTER NOTICES.

Lezioni sulle Teoria delle Superficie. By GREGORIO RICCI. Padova, 1898.

THIS lithographed volume of about 400 pages gives the formal development and application to the theory of surfaces of the instrument of analysis termed Calcolo differenziale assoluto by the author. The method leads to formulæ and equations always presenting themselves under the same form for any system of independent variables, and the difficulties which are incidental and formal rather than intrinsic are thus to some extent done away with, and the research assumes a uniformity absent in other methods. The entire discussion is based on the properties of differential quadratic forms, and the Introduction, of about 130 pages, contains a rather complete exposition of the methods of the calculus with a more or less full treatment of differential quadratic forms in general and binary forms in particular. The novelty and generality of the method and notation require of the reader not a little effort; but, these difficulties of form once overcome, the various geometrical applications follow easily.

Chapter I is devoted to generalities on linear homogeneous partial differential equations of the first order and to complete systems. Chapters II and III treat differential quadratic forms and develop the absolute calculus for n variables. In chapters IV and V these forms are classified and differential invariants are formed, while chapter VI applies the calculus to two variables.

Part I, comprising the second third of the book, treats surfaces as flexible and inextensible, while Part II is taken up with the theory of surfaces considered as having a rigid form in space. Chapter IV of this part treats special classes of surfaces, while chapter VI is devoted to surfaces of the second degree.

The treatise is by no means a complete exposition of the whole theory of surfaces, such subjects as infinitesmal deformations, pseudospherical geometry, and triply orthogonal systems being barely mentioned; but the idea of the volume seems to be to give some notion of the power and elegance of the absolute calculus. This has a wider application than that indicated here. many questions of pure mathematics and of mathematical physics being advantageously treated by it; and those accustomed to the classic method and notation, as found in Darboux and Bianchi for instance, will

find the book well worth reading if for no other reason than to look at the theory from another and entirely different aspect.

G. O. JAMES.

Leçons sur la Théorie des Formes et la Géométrie analytique supérieure. Par H. ANDOYER. Volume I. Paris, Gauthier-Villars, 1900. 8vo, pp. vi + 508. Price, 15 francs.

IT would be asserting too much to say that M. Andoyer's first volume fulfills all reasonable expectations. In this volume only binary and ternary forms are treated, the quaternary field being reserved for a second, which is announced as already in press. All who read the author's introduction (lithographed) a few years ago must have expected to find in the present work a treatise not only compendious but also elementary. Compendious it certainly is, covering a surprisingly wide range, but the student beginning in this subject and reading it alone will find it impenetrable. It is lucid, it is concise, but it is extremely condensed. Therein lies, however, its great merit. As a work of reference, or as a syllabus to accompany a lecture course, it will supersede anything hitherto published in the same field. Its field is geometry—invariantive geometry, algebra taking the second place. Accordingly one cannot yet dispense with Salmon, Faa di Bruno, and Elliott. Nor is it intended to precede the study of projective geometry of curves; rather it presupposes a large amount of geometrical knowledge, and aims to recast and systematize it. To quote from the preface: "Je me suis proposé, en l'écrivant, d'exposer d'une façon didactique la théorie des formes et son interprétation géométrique générale."

Binary forms are treated in ten chapters, occupying 145 pages. After two excellent but too compact preliminary chapters, linear and quadratic forms and systems of forms are fully discussed, together with formal treatment of resultants and discriminants. Cubics, quartics, and quintics with their full covariant systems are given, the last only in list without any detail. All systematic discussion of *complete* form systems is excluded, the reader being referred to Gordan and Hilbert. Finally forms in two sets of variables are taken up, otherwise correspondences, and the metric geometry on a line. The chapter most novel in this binary division is that on the doubly quadratic form, or the (2, 2) correspondence on a line. Of course the problem of derived correspondences (2, 2) and the conditions for the occurrence

of closed systems are of principal importance here, and I believe that no algebraic treatment has been accessible in textbooks heretofore. Andoyer derives the explicit fundamental invariants and covariants, the recursive formula expressing the equation of the $(n+1)$th derivative correspondence in terms of the nth and $(n-1)$th, and so ultimately in rational covariants of the original form. The results here obtained are directly applied in the chapter on Poncelet's polygons, in the ternary division.

Under ternary forms occur the necessary formal generalities, short chapters on homography and on conics (*la série quadratique*); then that just now alluded to upon the system of two conics, a welcome addition to the literature of the subject. Invariant conditions for closed Poncelet polygons of 3 and of 4 sides are given in rational functions of the four fundamental invariants, and the apparatus is furnished for calculating similar conditions for n sides. The derivation of these results by the aid of the elementary invariants of a binary biquadratic form is a decided improvement upon older methods.

Two bilinear forms and the quadratic correspondence are handled thoroughly, with a view to the quartic curve. Cubic and trilinear forms have a fairly full treatment (53 pages). The quartic curve is considered as the envelope of a variable conic whose parametric point describes a fixed conic: thus the bitangents and quadruply tangent conics are examined by the method of Cayley and Salmon. The two concluding chapters, apparently valuable, are concerned with metric geometry, general and special, *i. e.*, with a proper quadric and a degenerate quadric respectively as the absolute.

The thoroughly systematic execution of the work perhaps excuses the absence of any index, and the geometrical purpose naturally excludes any account of the work of Hilbert or of Deruyts; but a work of this magnitude, not professedly a cyclopædia, ought certainly to offer some perspectives and some outlook. The promise of a general bibliography in Volume II is hardly a substitute for the minute and particular citations that a student needs.—Enough of fault-finding! It is refreshing to read a treatise that makes perfectly free use of invariants of every degree of intricacy, with an eye solely to their significance and no concern about their explicit formulation. Let us notice that this last product of the nineteenth century on plane curve theory stops short of the quintic.

HENRY S. WHITE.

NOTES.

THE second (April) number of Volume II of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Canonical forms of quaternary abelian substitutions in an arbitrary Galois field," by L. E. DICKSON; "Certain cases in which the vanishing of the Wronskian is a sufficient condition for linear dependence," by M. BÔCHER; "An elementary proof of a theorem of Sturm," by M. BÔCHER; "On the determination of surfaces capable of conformal representation upon the plane in such a manner that geodetic lines are represented by algebraic curves," by H. F. STECKER; "On the existence of a minimum of the integral $\int_{x_0}^{x_1} F(x, y, y')dx$ when x_0 and x_1 are conjugate points, and the geodesics on an ellipsoid of revolution; a revision of a theorem of Kneser's," by W. F. OSGOOD; "On the geometry of planes in a parabolic space of four dimensions," by I. STRINGHAM.

THE April number (second series, volume 2, number 3) of the *Annals of Mathematics* contains the following papers: "Sufficient conditions in the calculus of variations," by W. F. OSGOOD; "Lagrange's equation in the calculus of variations and the extension of a theorem of Erdmann," by J. K. WHITTEMORE; "On some points in the theory of the hypergeometric function, expressed as a double circuit integral," by R. M. HATHAWAY; "A theorem in continued fractions," by D. N. LEHMER; "Note on the dual of a focal property of the inscribed ellipse," by R. E. ALLARDICE; "A simplified solution of the cubic," by EMORY MCCLINTOCK.

AT the regular meeting of the London mathematical society held on March 14, 1901, the following papers were read: "An account of some algebraical identities of simple arithmetical application," by Professor E. B. ELLIOTT; "Preliminary notice concerning the stability of motion," by Professor A. E. H. LOVE; "On the composition of group characteristics," by Professor W. BURNSIDE; "On the use of Cauchy's principal values in the double limit problems of the integral calculus," by Mr. G H. HARDY.

THE following papers were read at the meeting of the Edinburgh mathematical society held on March 8, 1901: "Some elementary theorems regarding surds," by Professor G. CHRYSTAL; "Note on the application of complex integra-

tion to the equations of conduction of heat," by Mr. J. DOUGALL.

THE first fasciculus of volume IV of the Encyclopaedia of the mathematical sciences (see *Bulletin*, volume 7, page 101) is to appear very shortly. It will contain "Vector analysis," by Dr. M. ABRAHAM of Göttingen, and Professor A. E. H. LOVE's two articles on "Hydrodynamics."

THE first fasciculus of volume 9 of the *Jahresberichte* of the German mathematical association has just appeared. It contains an announcement that volume 10 will include a report by Professor H. BURKHARDT entitled: "Die Ausbildung der Methode der Reihenentwickelungen an physikalischen Problemen."

THE Spanish journal *El Progreso matematico*, edited by D. ZOEL G. DE GALDEANO, Saragossa, ceased publication with the number for last December.

THE first two numbers of a new Italian monthly, *Le Matematiche pure ed applicate* have appeared. It is edited by Professor C. ALASIA of Oristano with the coöperation of twenty Italian and foreign mathematicians, and is published by S. Lapi in Città di Castello. The periodical will publish notes and memoirs on elementary and advanced mathematics, but preference will be given to articles on applied mathematics. The first number, which has twenty-four pages, begins with an article on continued fractions by the late Professor C. HERMITE. Contributions may be sent to the American member of the board of editors, Professor G. B. HALSTED, Austin, Texas.

THE announcement is made that Professor A. VON OETTINGEN of Leipsic has undertaken to edit Volume IV. of Poggendorff's Biographical Dictionary, which is to cover the period 1884–1900. Authors of works and articles in mathematics, physics, astronomy, meteorology, chemistry, mineralogy, geology, and geography have been asked to promote this undertaking by sending to Professor v. Oettingen biographical notices, and also information regarding books and articles published in periodicals which are not very widely known. The new volume will appear in 1902 or 1903, and will be published by J. A. Barth, of Leipsic.

CAMBRIDGE UNIVERSITY. The Smith's prizes have been adjudged to G. H. HARDY, fourth wrangler in 1898, for his essay on "Definite integrals of discontinuous functions,"

and to J. H. JEANS, bracketed second wrangler in 1898, for his essay on "The distribution of molecular energy." P. V. BEVAN, fourth wrangler in 1899, received an honorable mention for his essay on "The influence of metallic media on light vibrations."

THE UNIVERSITY OF CHICAGO.—The following advanced mathematical courses, four hours weekly, are to be offered during the four quarters (*su*, *a*, *w*, *sp*) of the year beginning June 19, 1901:—By Professor E. H. MOORE: Theory of functions of a real variable, with seminar (*a*, *w*, *sp*).—By Professor O. BOLZA: Theory of functions of a complex variable (*su*); Calculus of variations (*su*); Advanced algebra (*a*, *w*); Abelian functions, with seminar (*a*, *w*).—By Associate Professor H. MASCHKE: Theory of functions (*a*, *w*); Theory of the potential (*sp*); Theory of invariants (*a*); Twisted curves and surfaces, with seminar (*w*, *sp*).—By Assistant Professor H. E. SLAUGHT: Differential equations (*su*); Advanced integral calculus (*a*, *w*, *sp*).—By Assistant Professor J. W. A. YOUNG: Pedagogy of mathematics (*su*); Solid analytics (*sp*).—By Assistant Professor L. E. DICKSON: Pure and analytic projective geometry (*su*); Theory of numbers, with introduction to congruence groups (*sp*).—By Dr. J. A. BOYD: Solid analytics and determinants (*su*).—By Dr. MCDONALD: Hyperelliptic functions (*su*).

COLUMBIA UNIVERSITY.—The following advanced courses are offered during the academic year 1901–1902 by the departments of mathematics and mechanics, each course extending throughout the year:—By Professor F. N. COLE: Theory of groups, three hours.—By Professor T. S. FISKE: Advanced calculus, three hours; Functions defined by linear differential equations, three hours.—By Professor J. MACLAY: Theory of functions of a complex variable, three hours.—By Professor M. I. PUPIN: Electromagnetic theory of light, two hours; Bessel's functions and spherical harmonics, one hour.—By Professor R. S. WOODWARD: Advanced theoretical mechanics, two hours; Theory of the potential function, two hours.—By Dr. E. KASNER: Theoretical mechanics, three hours.—By Mr. C. J. KEYSER: Modern theories of geometry, three hours.—By Mr. H. B. MITCHELL: Differential equations, three hours.—By Mr. J. C. PFISTER: Theoretical mechanics, two hours.

The mathematical colloquium is held fortnightly.

CORNELL UNIVERSITY.—Among the courses announced for the summer session, July 5th to August 16th, are ele-

mentary courses in algebra, geometry, trigonometry, analytic geometry, and infinitesimal calculus, given by Professors TANNER, JONES, WAIT, and Dr. MURRAY, and the following advanced courses:—By Professor L. A. WAIT: Analytic geometry, based on Salmon's conic sections, three hours; Differential calculus, based on the treatises of Todhunter and Williamson, two hours.—By Dr. V. SNYDER: Projective geometry, an elementary course in Cremona's Projective geometry, five hours.—By Dr. D. A. MURRAY: Differential equations, five hours; Descriptive astronomy, five hours.—By Dr. G. A. MILLER: Integral calculus, based on the treatises of Todhunter and Williamson, five hours; Introduction to the theory of groups and the theory of numbers, five hours; History of mathematics, five hours.—By Dr. J. V. WESTFALL: Theory of functions of a complex variable, the elements of the theories of Cauchy, Riemann, and Weierstrass, three hours.

HARVARD UNIVERSITY.—The following advanced mathematical courses are offered during the academic year 1901–1902:—By Professor J. M. PEIRCE: Quaternions (first course); Triangular coördinates and algebraic plane curves, especially cubics; † Application of quaternions to the theory of curves and surfaces; † Selected topics in quaternions.—By Professors W. E. BYERLY and B. O. PEIRCE: Trigonometric series, spherical harmonics, and potential function.—By Professor W. E. BYERLY: Calculus (second course); Dynamics of a rigid body.—By Professor W. F. OSGOOD: † Infinite series and products; Theory of functions (first course).—By Professor M. BÔCHER: † Algebra, properties of polynomials, invariants; † Introduction to partial differential equations; Functions defined by linear differential equations.—By Dr. C. L. BOUTON: † Theory of numbers; † Elementary differential equations.—By Mr. J. K. WHITTEMORE: Modern geometry; Hydrostatics, hydrokinetics.—By Mr. J. L. COOLIDGE: † Theory of equations, invariants; Non-euclidean geometry.

These courses will involve three lectures a week throughout the year, except those preceded by †, which involve about half this number of lectures. Professors BYERLY and OSGOOD, Dr. BOUTON, and Mr. COOLIDGE also offer courses in reading and research on Picard's Traité d'analyse, Calculus of variations, Theory of continuous groups, and Projective geometry, respectively.

The mathematical conference will meet twice a month.

YALE UNIVERSITY.—The following courses in mathematics

are announced for the year 1901–1902 :—By Professor J. E. CLARK : † Determinants, with applications to geometry and the theory of elimination, two hours ; † Differential equations, two hours.—By Professor J. W. GIBBS ; † Vector analysis, three hours ; † Advanced vector analysis, three hours ; Electricity and magnetism, one hour ; Thermodynamics and properties of matter, two hours.—By Professor W. BEEBE : Celestial mechanics, three hours.—By Professor J. PIERPONT : † Higher algebra, three hours ; Differential equations and function theory, three hours; Theory of functions, three hours.—By Professor P. F. SMITH : Advanced differential geometry, two hours ; † Foundations of geometry, two hours.—By Professor H. A. BUMSTEAD : Problems in mathematical physics, two hours.—By Dr. M. B. PORTER : Advanced calculus, three hours ; † Selected topics in differential equations, three hours.—By Dr. W. A. GRANVILLE : † Differential geometry, three hours.—By Mr. E. B. WILSON : † Analytical mechanics, three hours ; † Projective geometry, three hours.

Courses preceded by † extend through only one half of the year.

THE several German universities below offer during the summer semester, 1901, courses in mathematics as follows :

UNIVERSITY OF BERLIN.—By Professor L. FUCHS : Applications of the theory of elliptic functions, three hours ; Theory of linear differential equations, four hours.—By Professor H. A. SCHWARZ : Integral calculus, four hours ; Theory of analytic functions, I, four hours ; Gauss's hypergeometric series, two hours ; Mathematical colloquium, two hours.—By Professor G. FROBENIUS : Theory of algebraic equations, II, four hours.—By Professor J. KNOBLAUCH : Theory of space curves, one hour ; Theory of curved surfaces, four hours ; Elliptic functions, four hours. By Professor K. HENSEL : Analytic geometry of plane and space, four hours ; Theory of probabilities, two hours ; Geometry of numbers and its application ; Colloquium, two hours.—By Professor R. HELMERT : Figure of the earth, one hour ; Map projection, one hour.—By Professor R. LEHMANN-FILHÉS : Differential Calculus, four hours, with exercises, one hour.—By Professor G. HETTNER : Fourier's series and integrals, two hours.

UNIVERSITY OF BONN.—By Professor L. HEFFTER : Infinitesimal calculus, four hours, with exercises, one hour ; Descriptive geometry, two hours, with exercises, three hours.—

By Professor R. LIPSCHITZ: Theory of numbers, four hours; Seminar, two hours.—By Professor H. KORTUM; Theory of algebraic equations, four hours; Seminar, two hours.

UNIVERSITY OF HALLE.—By Professor G. CANTOR: Theory of numbers, four hours; Definite integrals and differential equations, two hours; Seminar, two hours.—By Professor A. WANGERIN: Theory of potential and spherical harmonics, five hours; Seminar, two hours.—By Professor V. EBERHARD: Theory of Determinants, one hour: Plane analytic geometry, three hours.—By Dr. E. NEUMANN: Infinitesimal calculus, I, four hours, with exercises, one hour.

UNIVERSITY OF INNSBRUCK.—By Professor O. STOLZ: Theory of double integrals, two hours; Calculus of variations, one hour; Seminar on the foregoing subjects, one hour; Theory of functions of a complex variable according to Cauchy and Weierstrass (continued), with seminar, three hours.—By Professor W. WIRTINGER: Linear partial differential equations (continued), three hours; Euler's and the hypergeometric integrals, two hours; Seminar, two hours.—By Dr. K. ZINDLER: Descriptive geometry, four hours; Applications of calculus to geometry (continued), two hours.

UNIVERSITY OF VIENNA.—By Professor J. v. ESCHERICH: Calculus of variations, five hours; Mathematical statistics, three hours; Proseminar, one hour; Seminar, two hours.—By Professor L. GEGENBAUER: Theory of quadratic forms, two hours; Theory of numbers, II, three hours; Theory of probabilities, three hours; Proseminar, one hour; Seminar, two hours.—By Professor F. MERTENS: Elements of infinitesimal calculus for students of all faculties, II, five hours, with exercises, two hours; Proseminar, one hour; Seminar, two hours.—By Professor G. KOHN: Synthetic geometry (continued), four hours, with exercises, one hour.—By Dr. V. SERSAWY: Mathematics of insurance, two courses of three and four hours respectively.—By Dr. A. TAUBER: Analytic geometry of space, three hours; Mathematics of insurance, three hours, with exercises, two hours.—By Dr. E. BLASCHKE: Introduction to mathematical statistics.—By Dr. R. D. v. STERNECK: On the number of prime numbers between given limits, one hour.—By Dr. J. A. GMEINER: Quaternions, with applications to the motion of points, one hour.

A COURSE of six lectures on "British mathematicians of the nineteenth century" was delivered by Dr. ALEXANDER

MACFARLANE at Lehigh University, April 12–23. The life and work of the following mathematicians was presented: George Peacock (1791–1858), Augustus De Morgan (1806–1871), W. R. Hamilton (1805–1865), George Boole (1815–1864), Arthur Cayley (1821–1895), W. K. Clifford (1845–1879).

DR. J. C. FIELDS has recently given at the University of Toronto a course of lectures on "The quadrature of the circle."

PROFESSOR J. W. GIBBS, of Yale University, has been elected an honorary member of the London physical society.

PROFESSOR G. HUMBERT of the Ecole polytechnique has been elected a member of the section of geometry of the Paris academy of sciences to fill the vacancy caused by the death of HERMITE.

PROFESSOR ASAPH HALL, U. S. Navy, who for the last five years has been lecturer on celestial mechanics at Harvard University, has resigned his position there, and will spend the next year or two in European travel.

IT is reported that the faculty of the University of Munich has nominated Professors D. HILBERT, A. BRILL, and A. Voss to fill the vacancy caused by the resignation of Professer G. BAUER.

DR. E. A. ENGLER, professor of mathematics in Washington University, St. Louis, Mo., has been elected president of the Worcester Polytechnic Institute, to succeed Dr. T. C. MENDENHALL, who retires on account of ill-health.

AT Columbia University, Dr. JAMES MACLAY has been promoted to an associate professorship of mathematics, and Mr. J. C. PFISTER to an instructorship in mechanics. Dr. G. H. LING, of Wesleyan University, and Mr. WILLIAM FINDLAY, of the University of Chicago, have been appointed tutors in mathematics in Columbia University.

MR. H. W. KUHN, of Cornell University, has been appointed assistant professor of mathematics in Ohio State University.

DR. R. E. MORITZ, of the University of Nebraska, has received leave of absence and will spend a year in study in Germany.

PROFESSOR H. A. ROWLAND, Johns Hopkins University, died at Baltimore, April 16.

THE death is announced of DR. J. T. DUFFIELD, professor of mathematics in Princeton University since 1847.

THE deaths are also announced of M. T. MOUTARD, an eminent French mathematician and engineer and honorary examiner at the Ecole polytechnique; of Dr. P. M. POKROWSKI, professor of mathematics at Kiew; and of Dr. F. MELDE, professor of physics and director of the mathematical and physical institute of the University of Marburg.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

COLLINS (J. V.). An elementary exposition of Grassmann's Ausdehnungslehre, or theory of extension. Springfield, Mo., Dixon, 1901, 8vo. 2+46 pp. (Reprinted from the *American Mathematical Monthly*, Vols. 6 and 7.)

KIEPERT (L.). Grundriss der Differential- und Integralrechnung. Teil I: Differentialrechnung. 9te Auflage des gleichnamigen Leitfadens von M. Stegemann. Hannover, Helwing, 1901. 8vo. 17 + 750 pp. Cloth. M. 12.00

MANNING (H. P.). Non-euclidean geometry. Boston, Ginn, 1901. 12mo. 4 + 95 pp. Cloth. $0.75

MOLK (J.). See TANNERY (J.).

NICHOLS (E. W.). Differential and integral calculus, with applications; for colleges, universities and technical schools. Boston, Heath, 1900. 12mo. 405 pp. Half-leather. $2.00

STEGEMANN (M.). See Kiepert (L.).

TANNERY (J.) et MOLK (J.). Eléments de la théorie des fonctions elliptiques. (En 4 volumes.) Vol. 4: Calcul intégral, partie 2e, et applications. Fascicule I. Paris, Gauthier-Villars, 1900. 8vo. Pp. 1—164. Vol. 4 complete Fr. 9.00

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